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Characterization of Smoothness Properties of Functions by Means of Their Degree of Approximation by Splines

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One of the major achievements of approximation theory is the following:

THEOREM 1. (D. Jackson [5], S. N. Bernstein [1], A. Zygmund [10]). *Let f be a real function with domain $(-\infty, \infty)$, and period 2π. Let q be a positive number, let p be the largest integer smaller than q, and set $\alpha = q - p$, so that $q = p + \alpha$, $0 < \alpha \leqslant 1$. A necessary and sufficient condition that there exist a constant A, and for $n = 1, 2,\ldots$ a real trigonometric polynomial*

$$f_n(x) \equiv a_0^{(n)} + \sum_{k=1}^{n} a_k^{(n)} \cos kx + b_k^{(n)} \sin kx \tag{1}$$

such that

$$\sup_{-\infty<x<\infty} |f(x) - f_n(x)| \leqslant A/n^q, \tag{2}$$

is

(i) *if q is not an integer, that $f^{(p)}$ exist throughout $(-\infty, \infty)$, and satisfy there a Lipschitz condition of order α;*

(ii) *if q is an integer, that $f^{(p)}$* (i.e., $f^{(q-1)}$) *exist and be continuous in $(-\infty, \infty)$, and satisfy, for some constant B,*

$$|f^{(p)}(x) - 2f^{(p)}(x+h) + f^{(p)}(x+2h)| \leqslant Bh \tag{3}$$

whenever $-\infty < x < x + 2h < \infty$.

The purpose of the present article is to prove a result of exactly the same form as Theorem 1 in which the trigonometric polynomials (1) are replaced by splines f_n, where n refers not to their degree but to the number of knots, the knots being equidistant. As to the degree k of the splines, we have

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complete freedom in its choice as long as $q < k + 1$. We include, however, also the case $q \geqslant k + 1$.

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THEOREM 2. *Let* $-\infty < a < b < \infty$, *and let* f *be a real function with domain* $[a, b]$. *Let* q *be a positive number,* k *an integer* $\geqslant 1$, *and define* p *and* α *as in Theorem* 1. *A necessary and sufficient condition that* $(*)$ *there exist a constant* A, *and for* $n = 1, 2, \ldots$ *a real function* f_n *which in each interval*

$$I_{n,j} = [a + (b - a)\, n^{-1}(j - 1),\, a + (b - a)\, n^{-1}j], \qquad j = 1, 2, \ldots, n,$$

coincides with a polynomial of degree $\leqslant k$, *and which has a* $(k - 1)$ th *derivative throughout* (a, b) *such that*

$$\sup_{a \leqslant x \leqslant b} |f(x) - f_n(x)| \leqslant A/n^q,$$

is

(i) *if* q *is not an integer and* $q < k + 1$, *or if* $q = k + 1$, *that* $f^{(p)}$ *exist*[1] *throughout* $[a, b]$ *and satisfy there a Lipschitz condition of order* α;

(ii) *if* q *is an integer* $< k + 1$, *that* $f^{(p)}$ (i.e., $f^{(q-1)}$) *exist and be continuous in* $[a, b]$, *and satisfy, for some constant* B, *the inequality* (3) *whenever* $a \leqslant x < x + 2h \leqslant b$;

(iii) *if* $q > k + 1$, *that* f *coincide in* $[a, b]$ *with some polynomial of degree* $\leqslant k$.

Proof. Necessity. If $q \leqslant k$, necessity follows from the Lemma below, since $\tilde{p}$ of the Lemma is $\leqslant k - 1$. If $k < q \leqslant k + 1$, it follows from Theorem 1 of [6]. Finally, if $q > k + 1$, it follows from [3] (Korollar, p. 130).

Sufficiency. We may clearly assume $q \leqslant k + 1$. Whether or not q is an integer, if $f^{(p)}$ exists throughout $[a, b]$ and satisfies there a Lipschitz condition of order α, then, by [2, Corollaries 1 and 2, p. 233], $(*)$ of Theorem 2 holds. Suppose now q is an integer $< k + 1$, $f^{(p)}$ exists and is continuous in $[a, b]$, and satisfies (3) whenever $a \leqslant x < x + 2h \leqslant b$. From Theorem 2 of [7] [(d) implies (a)] it follows that, for $n = 1, 2, \ldots$, there exists a real function F_n which in each interval $I_{n,j}$, $j = 1, 2, \ldots, n$, coincides with a polynomial of degree $\leqslant k - p$, and which has a $(k - p - 1)$ th derivative throughout (a, b) such that $\max_{a \leqslant x \leqslant b} |f^{(p)}(x) - F_n(x)| \leqslant A^* n^{-1}$, A^* being a constant. If $p = 0$, we are through; hence assume $p > 0$. For $n = 1, 2, \ldots$, let G_n be a real

[1] If $p \geqslant 1$, then $f^{(p)}$ at a and at b, the end points of the domain of f, is understood to be a one sided pth derivative. Similarly below.

function with domain $[a, b]$ which in each $I_{n,j}$, $j = 1, 2,..., n$, coincides with a polynomial of degree $\leqslant k$, and for which $G_n^{(p)} = F_n(x)$ throughout (a, b). Then $|f^{(p-1)}(y) - G_n^{(p-1)}(y) - \{f^{(p-1)}(x) - G_n^{(p-1)}(x)\}| \leqslant A^* n^{-1}(y - x)$ whenever $a \leqslant x < y \leqslant b$. By the above corollaries in [2], for $n = 1, 2,...$ there is a real function g_n with domain $[a, b]$, which in every $I_{n,j}$, $j = 1, 2,..., n$, coincides with a polynomial of degree $\leqslant k$ and which has a $(k - 1)$ th derivative throughout (a, b) such that $\max_{a \leqslant x \leqslant b} |f(x) - G_n(x) - g_n(x)| \leqslant A/n^{p+1}$, A being a constant (*independent of* n). We can now take $f_n = G_n + g_n$, $n = 1, 2,...$. This completes the proof.

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LEMMA. *Let f be a real function with domain $[a, b]$ $(-\infty < a < b < \infty)$, let k and p be integers $\geqslant 0$, and let $0 < \alpha \leqslant 1$. If $\alpha = 1$ and $k = p + 1$, let N be the set of all numbers $2^u 3^v$, $u = 0, 1, 2,...$, $v = 0, 1, 2,...$. Otherwise, let $N = \{2^0, 2^1, 2^2,...\}$. If $\alpha = 1$ and $k \neq p + 1$, let $\tilde{p} = p + 1$. Otherwise, let $\tilde{p} = p$. Suppose there is a constant A, and for every $n \in N$ a real function f_n with domain $[a, b]$ for which $f_n^{(\tilde{p})}$ exists throughout (a, b), which in each interval $I_{n,j} = [a + (b - a)\, n^{-1}(j - 1), a + (b - a)\, n^{-1} j]$, $j = 1, 2,..., n$, coincides with a polynomial of degree $\leqslant k$ such that*

$$\sup_{a \leqslant x \leqslant b} |f(x) - f_n(x)| \leqslant A n^{-(p+\alpha)}.$$

Then $f^{(p)}$ exists throughout $[a, b]$. If $\alpha < 1$, $f^{(p)}$ satisfies there a Lipschitz condition of order α. If $\alpha = 1$, $f^{(p)}$ is continuous in $[a, b]$, and there is a constant B such that $|f^{(p)}(x) - 2f^{(p)}(x + h) + f^{(p)}(x + 2h)| \leqslant Bh$ whenever

$$a \leqslant x < x + 2h \leqslant b.$$

Proof. If $\alpha = 1$ and $k = p + 1$, let d be an integer $\geqslant 0$. Otherwise, set $d = 0$. Let $V_{0,d}(x) \equiv f_{3^d}(x)$, and for $\nu = 1, 2,...$, let

$$V_{\nu,d}(x) \equiv f_{2^\nu 3^d}(x) - f_{2^{\nu-1} 3^d}(x).$$

Then $\sum_{\nu=0}^{\infty} V_{\nu,d}(x)$ converges to $f(x)$ in $[a, b]$. Also, using the sup norm over $[a, b]$,

$$\| V_{\nu,d} \| \leqslant \| f_{2^\nu 3^d} - f \| + \| f - f_{2^{\nu-1} 3^d} \| \leqslant C(2^\nu 3^d)^{-(p+\alpha)},$$
$$\nu = 1, 2,...; \qquad C = A(1 + 2^{p+\alpha}). \tag{4}$$

Using W. A. Markoff's inequality [9, p. 36] in each of the intervals $I_{2^\nu 3^d, j}$, $j = 1, 2,..., 2^\nu 3^d$, we obtain, for $\nu = 1, 2,...,$ and $h = 0, 1,..., p$,

$$\| V_{\nu,d}^{(h)} \| \leqslant [2^{\nu+1}3^d/(b-a)]^h \| V_{\nu,d} \| \prod_{j=0}^{h-1} (k^2 - j^2)/(2j+1)$$

$$\leqslant C[2/(b-a)]^h \left[\prod_{j=0}^{h-1} (k^2 - j^2)/(2j+1) \right] (2^\nu 3^d)^{h-p-\alpha}, \qquad (5)$$

where $\prod_{j=0}^{h-1}$ means 1 if $h = 0$. Therefore, for $h = 0, 1,..., p$, $\sum_{\nu=0}^{\infty} \| V_{\nu,d}^{(h)} \|$ converges, and therefore $\sum_{\nu=0}^{\infty} V_{\nu,d}^{(h)}(x)$ converges uniformly in $[a, b]$. Hence throughout $[a, b]$, $f^{(p)}$ exists and equals $\sum_{\nu=0}^{\infty} V_{\nu,d}^{(p)}(x)$. Let D denote the coefficient of $(2^\nu 3^d)^{h-p-\alpha}$ in the extreme right member of (5), for $h = p$, and let $D^* = D2^{-\alpha}(1 - 2^{-\alpha})$. Then throughout $[a, b]$, for $m = 0, 1, 2,...,$ $| f^{(p)}(x) - \sum_{\nu=0}^{m} V_{\nu,d}^{(p)}(x)| \leqslant \sum_{\nu=m+1}^{\infty} \| V_{\nu,d}^{(p)} \| \leqslant D^*(2^m 3^d)^{-\alpha}$. In particular, if $f^{(p)}(x) \not\equiv 0$, as we can and shall assume, $k \geqslant p$. Given $n \in N$, say $n = 2^m 3^\delta$, m and δ nonnegative integers, set $V_n(x) \equiv \sum_{\nu=0}^{m} V_{\nu,\delta}^{(p)}(x)$, and observe that

> in each interval $I_{n,j}$, $j = 1, 2,..., n$, $V_n(x)$ coincides with a polynomial of degree $\leqslant k - p$; $\sup_{a \leqslant x \leqslant b} | f^{(p)}(x) - V_n(x)| \leqslant D^* n^{-\alpha}$, and, so, $f^{(p)}$ is continuous in $[a, b]$; if $\alpha = 1$ and $k \neq p + 1$, then $V_n(x)$ is differentiable throughout (a, b). (6)

We set $U_\nu(t) \equiv V_{\nu,0}(t)$, $\nu = 0, 1, 2,....$

(A) Suppose $\alpha < 1, p = 0$, so that $d = 0$. Let $a \leqslant x < y \leqslant b$, and let $\tilde{n}$ be the smallest positive integer n satisfying $2^n(y - x) \geqslant b - a$. We have:

$$| f(y) - f(x)| = \left| \sum_{\nu=0}^{\infty} U_\nu(y) - U_\nu(x) \right| \leqslant \sum_{\nu=0}^{\tilde{n}-1} | U_\nu(y) - U_\nu(x)| + 2 \sum_{\nu=\tilde{n}}^{\infty} \| U_\nu \|. \qquad (7)$$

Let $\nu \geqslant 0$. We show

$$| U_\nu(y) - U_\nu(x)| \leqslant 2^{\nu+1}k^2(b-a)^{-1} \| U_\nu \|(y - x). \qquad (8)$$

Set

$$x_j = a + (b - a)\, 2^{-\nu}j, \qquad j = 0, 1,..., 2^\nu. \qquad (9)$$

(a) Assume both x and y belong to some interval $[x_{j-1}, x_j]$, $1 \leqslant j \leqslant 2^\nu$. By A. A. Markoff's inequality [9, p. 36] applied to that interval,

$$| U_\nu(y) - U_\nu(x)| = (y - x)| U_\nu'(z)| \leqslant 2^{\nu+1}k^2(b-a)^{-1} \| U_\nu \|(y - x), \ \ x < z < y.$$

(b) Suppose the assumption in (a) does not hold. Let

$$x_{r-1} \leqslant x < x_r \cdots \leqslant x_{r+s} < y \leqslant x_{r+s+1}, \quad 1 \leqslant r < 2^\nu, \quad s \geqslant 0.$$

Then (with $\sum_{j=1}^{s}$ meaning 0 if $s = 0$), we have by A. A. Markoff's inequality,

$$\begin{aligned} &| U_\nu(y) - U_\nu(x)| \\ &\leqslant | U_\nu(y) - U_\nu(x_{r+s})| + \left[\sum_{j=1}^{s} | U_\nu(x_{r+j}) - U_\nu(x_{r+j-1})|\right] + | U_\nu(x_r) - U_\nu(x)| \\ &\leqslant 2^{\nu+1}k^2(b-a)^{-1} \| U_\nu \| \left[y - x_{r+s} + \left\{\sum_{j=1}^{s} x_{r+j} - x_{r+j-1}\right\} + x_r - x\right] \\ &= 2^{\nu+1}k^2(b-a)^{-1} \| U_\nu \| (y-x). \end{aligned}$$

Setting $E = \max(C, \|f_1\|)$, we have from (7), (8) and (4):

$$\begin{aligned} &|f(y) - f(x)| \\ &\leqslant Ek^2(b-a)^{-1}(y-x) \sum_{\nu=0}^{\tilde{n}-1} 2^{\nu+1}2^{-\alpha\nu} + 2E \sum_{\nu=\tilde{n}}^{\infty} 2^{-\alpha\nu} \\ &= 2Ek^2(b-a)^{-1}(y-x)[2^{(1-\alpha)\tilde{n}} - 1][2^{1-\alpha} - 1]^{-1} + 2E2^{-\alpha\tilde{n}}(1 - 2^{-\alpha})^{-1} \\ &\leqslant F[(y-x)\, 2^{(1-\alpha)\tilde{n}} + 2^{-\alpha\tilde{n}}], \end{aligned}$$

where $F = \max\,[2Ek^2(b-a)^{-1}(2^{1-\alpha} - 1)^{-1}, 2E(1 - 2^{-\alpha})^{-1}]$.

By definition of $\tilde{n}$, $2^{\tilde{n}}(y-x) \geqslant b - a$, $2^{\tilde{n}}(y-x) \leqslant 2(b-a)$. So $(y-x)^{-\alpha}[(y-x)\,2^{(1-\alpha)\tilde{n}} + 2^{-\alpha\tilde{n}}] \leqslant \{2(b-a)\}^{1-\alpha} + (b-a)^{-\alpha}$. Hence $|f(y) - f(x)| \leqslant L(y-x)^\alpha$, with $L = F[\{2(b-a)\}^{1-\alpha} + (b-a)^{-\alpha}]$.

(B) Suppose $\alpha < 1$, $p > 0$. By (6) and by part (A) applied to $f^{(p)}$, the latter satisfies in $[a, b]$ a Lipschitz condition of order α.

(C) Suppose $\alpha = 1$, $p = 0$, $k \neq 1$. Let $a \leqslant x < x + 2h \leqslant b$, and let $\tilde{m}$ be the largest positive integer n satisfying $2^n h \leqslant b - a$. We have

$$\begin{aligned} &|f(x) - 2f(x+h) + f(x+2h)| \\ &= \left| \sum_{\nu=0}^{\infty} U_\nu(x) - 2U_\nu(x+h) + U_\nu(x+2h) \right| \\ &\leqslant \sum_{\nu=0}^{\tilde{m}-1} | U_\nu(x) - 2U_\nu(x+h) + U_\nu(x+2h)| + 4 \sum_{\nu=\tilde{m}}^{\infty} \| U_\nu \|. \qquad (10) \end{aligned}$$

Let $\nu \geqslant 0$. We show

$$| U_\nu(x) - 2U_\nu(x+h) + U_\nu(x+2h)| \leqslant Gh^2 2^\nu, \quad G = (8/3)\, E(b-a)^{-2}\, k^2(k^2-1). \tag{11}$$

Now

$$\begin{aligned} U &= U_\nu(x) - 2U_\nu(x+h) + U_\nu(x+2h) \\ &= U_\nu(x+2h) - U_\nu(x+h) - \{U_\nu(x+h) - U_\nu(x)\} \\ &= h(U_\nu'(z) - U_\nu'(y)), \end{aligned}$$

where $x < y < x + h < z < x + 2h$. If [using the notation (9)] both y and z lie in some $[x_{j-1}, x_j]$, $1 \leqslant j \leqslant 2^\nu$, then $U = h(z-y)\, U_\nu''(w)$, $y < w < z$, and by W. A. Markoff's inequality applied to $[x_{j-1}, x_j]$,

$$| U_\nu''(w)| \leqslant 2^{2\nu+2}(b-a)^{-2}\, 3^{-1}k^2(k^2-1)\, \| U_\nu \|, \tag{12}$$

which, by (4), implies (11). If y and z do not belong to the same $[x_{j-1}, x_j]$, let $x_{r-1} \leqslant y < x_r \cdots \leqslant x_{r+s} < z \leqslant x_{r+s+1}$, $1 \leqslant r < 2^\nu$, $s \geqslant 0$. By (4) and inequalities similar to (12) we get again (11), since

$$\begin{aligned} &| U_\nu'(z) - U_\nu'(y)| \\ &\leqslant | U_\nu'(z) - U_\nu'(x_{r+s})| + \left[\sum_{j=1}^{s} | U_\nu'(x_{r+j}) - U_\nu'(x_{r+j-1})|\right] + | U_\nu'(x_r) - U_\nu'(y)| \\ &\leqslant (4/3)\, E(b-a)^{-2}\, k^2(k^2-1)\, 2^\nu \left[z - x_{r+s} + \left\{\sum_{j=1}^{s} x_{r+j} - x_{r+j-1}\right\} + x_r - y\right] \\ &\leqslant Gh2^\nu. \end{aligned}$$

By (10), (11) and (4), $| f(x) - 2f(x+h) + f(x+2h)| \leqslant Gh^2 2^{\bar{m}} + 8C2^{-\bar{m}}$. Now $(b-a)/(2h) < 2^{\bar{m}} \leqslant (b-a)/h$, and hence

$$| f(x) - 2f(x+h) + f(x+2h)| \leqslant [G(b-a) + 16C(b-a)^{-1}]\, h.$$

(D) More generally, suppose $\alpha = 1$, $k \neq p + 1$. In view of (6), we can apply the Lemma to $f^{(p)}$, with k replaced by $k - p$ and with p of the Lemma taken as 0, and obtain the desired conclusion.

(E) Suppose $\alpha = 1$, $p = 0$, $k = 1$. The method used in [8], p. 397, to prove sufficiency, clearly establishes also the Lemma in the present case. Namely, let $a \leqslant x < x + 2h \leqslant b$, and let n_0 be the largest $n \in N$ (i.e., n of the form $2^u 3^v$; $u = 0, 1, \ldots$, $v = 0, 1, \ldots$) for which $[x, x + 2h]$ is contained in some $I_{n,j}$, $1 \leqslant j \leqslant n$. Then $2h > (b-a)(6n_0)^{-1}$. For, otherwise, if, say, $[x, x+2h] \subseteq I_{n_0, j_0}$, $1 \leqslant j_0 \leqslant n_0$, then $[x, x + 2h]$ would lie either in one of the two closed halves of I_{n_0,j_0} or in the (open) middle third of I_{n_0,j_0}. In each case, the maximality of n_0 is contradicted. By the linearity of f_{n_0} in $[x, x + 2h]$, we have

$$\begin{aligned} &|f(x) - 2f(x + h) + f(x + 2h)| \\ &\quad = |\{f(x) - f_{n_0}(x)\} - 2\{f(x + h) - f_{n_0}(x + h)\} \\ &\qquad + \{f(x + 2h) - f_{n_0}(x + 2h)\}| \\ &\quad \leqslant 4A/n_0 \leqslant 48A(b - a)^{-1} h. \end{aligned}$$

(F) Suppose, finally, $\alpha = 1$, $k = p + 1$. In view of (6), we can apply the Lemma to $f^{(p)}$, with k replaced by $k - p$ and with p of the Lemma taken as 0.

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